IGCSE Additional Mathematics (0606):

Calculus

Lesson 14.03

Application of Differentiation (Part - 1)

Revision Notes

Cambridge will assess your ability to:

- Use differentiation to gradients, tangents and normals, stationary points
- Use the first and second derivative tests to discriminate between maxima and minima

1. Find the gradient of a function $y = x^2 - 2x - 3$ when $x =- 1$.

First find the derivative of $y = x^2 - 2x - 3$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 2x - 3) = 2x - 2$

$= 2(-1) - 2$

$=- 2 - 2 = \ - 4$

$\therefore$ The gradient of $y = x^2 - 2x - 3$ is -4 when $x =- 1$

2. Find the equation of the tangent line of the curve $y = x^2 - 2x - 3$ at $x = \ - 1$.

The general equation for a straight line is $y = mx + c$

Differentiating $y = x^2 - 2x - 3$ with respect to x, we get;

$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 2x - 3) = 2x - 2$

$= 2(-1) - 2$

$=- 2 - 2 = \ - 4$

$\therefore$ The gradient of $y = x^2 - 2x - 3$ =gradient of the tangent = -4 at $x =- 1$

i.e. $m =- 4$

So, $y =- 4x + c$

When $x = \ - 1$ then $y = x^2 - 2x - 3 = (-1)^2 - 2(-1) - 3 = 0$

The point of intersection is $(-1, 0)$.

Now, $y =- 4x + c$

when $x = \ - 1,\ y = 0$

then $0 =- 4(-1) + c$

$c = \ - 4$

The equation of the tangent is $y = 4x - 4$

3. Find the gradient and the equation of the normal to the curve

$y = x^2 - 2x - 3$ **at** $x = -1$.

First, we have to find the gradient of the tangent of the curve $y = x^2 - 2x - 3$ at $x = -1$

$\frac{dy}{dx} = \frac{d}{dx}(x^2 - 2x - 3) = 2x - 2$

$= 2(-1) - 2$

$= -2 - 2 = -4$

$\therefore$ The gradient of $y = x^2 - 2x - 3$ =gradient of the tangent = -4 at $x = -1$

Normal and tangent are perpendicular to each other.

So, $-4 \times m = -1$

$m = \frac{1}{4}$ (gradient of the normal to the curve at $x = -1$)

The equation of the normal is given by $y = mx + c$

Given $m = \frac{1}{4}$ and the point of intersection $x = -1,\ y = 0$

$0 = \frac{1}{4}(-1) + c$

$c = \frac{1}{4}$

Therefore, the equation of normal line:

$y = \frac{1}{4}x + \frac{1}{4}$

4. Find the stationary point of the curve $y = x^2 - 2x - 3$.

Here, $\frac{dy}{dx} = 2x - 2$

Set $\frac{dy}{dx} = 0$

$2x - 2 = 0$

$x = 1$

When $x = 1$, then $y = (1)^2 - 2(1) - 3 = -4$

Hence, there is a stationary point at $(1, -4)$ on the curve $y = x^2 - 2x - 3$.

5. Find the equation of the tangent and normal to the curve

$y = (x^2 - 2)(3 - x)$ at the point $(-1, -4)$ and the stationary point of the curve.

$\frac{dy}{dx} = \left(x^2 - 2\right)(-1) + (3 - x)(2x)$

$= 2 + 6x - 3x^2$

At point $(-1, -4)$:

Gradient of the tangent: $2 + 6(-1) - 3(-1)^2 = -7$

Equation: $y - (-4) = -7(x - (-1))$

$y = -7x - 11$

Gradient of the normal: $-7 \times m = -1$

$$m = \frac{1}{7}$$

Equation of the normal: $y - (-4) = \frac{1}{7}(x - (-1))$

$$7y = x - 27$$

Stationary point:

$\frac{dy}{dx} = 2 + 6x - 3x^2$

$\frac{dy}{dx} = 0$

$2 + 6x - 3x^2 = 0$

Using the quadratic formula to solve the quadratic equation $-3x^2 + 6x + 2 = 0$

Here, $a = -3,\ b = 6,\ c = 2$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(6) \pm \sqrt{(6)^2 - 4 \times (-3) \times (2)}}{2(-3)}$

$x = \frac{-6 \pm \sqrt{60}}{-6}$

$x = 1 + \frac{-1}{3}\sqrt{15}$ or $x = 1 + \frac{1}{3}\sqrt{15}$

When $x = 2.29$ then $y = 2.30$

When $x = -0.29$ then $y = -6.30$

Therefore, the two stationary points are $(2.29, 2.30)$ and $(-0.29, -6.30)$

6. Find the turning points of the graph of the function,

$y = x^3 + 5x^2 - 8x + 2$ and determine whether each turning point is a maximum or minimum point.

$y = x^3 + 5x^2 - 8x + 2$

$\frac{dy}{dx} = 3x^2 + 10x - 8$

At the turning points,

$\frac{dy}{dx} = 0$

$3x^2 + 10x - 8 = 0$ This implies, $x = -4, \frac{2}{3}$

When $x = -4$ then $y = (-4)^3 + 5(-4)^2 - 8(-4) + 2 = 50$

When $x = \frac{2}{3}$ then $y = (\frac{2}{3})^3 + 5(\frac{2}{3})^2 - 8(\frac{2}{3}) + 2 = -\frac{22}{27}$

The turning points are: $(-4, 50)$ and $\left(\frac{2}{3}, -\frac{22}{27}\right)$

To find the nature of turning points, let's find the second derivative of $\frac{dy}{dx}$;

$\frac{d^2y}{dx^2} = 6x + 10$

When $x = -4$, then $\frac{d^2y}{dx^2} = 6x + 10 = 6(-4) + 10 = -14$ (negative)

$\therefore$ $(-4, 50)$ is a maximum point.

When $x = \frac{2}{3}$, then $\frac{d^2y}{dx^2} = 6x + 10 = 6(\frac{2}{3}) + 10 = 14$ (positive)

$\therefore$ $(\frac{2}{3}, -\frac{22}{27})$ is a minimum point.

Alternatively, We can also determine the nature of turning points by first derivative test;

For turning point $(-4, 50)$; checking the first derivative value of curve on left and right side of the turning point, we get:

$\frac{dy}{dx} = f'(-5) = 3(-5)^2 + 10(-5) - 8 = 17$ (positive)

$\frac{dy}{dx} = f'(-3) = 3(-3)^2 + 10(-3) - 8 = -11$ (negative)

Hence the gradient of the curve is decreasing around the turning point $(-4, 50)$, so it is the maximum point.

Likewise ; for turning point $(\frac{2}{3}, -\frac{22}{27})$

$\frac{dy}{dx} = f'(0) = 3(0)^2 + 10(0) - 8 = -8$ (positive)

$\frac{dy}{dx} = f'(1) = 3(1)^2 + 10(1) - 8 = 5$ (negative)

Hence the gradient of the curve is increasing around the turning point $(-4, 50)$, so it is the minimum point.